THE SCOTS COLLEGE

YEAR 12

HSC TRIAL EXAMINATION

MATHEMATICS - EXTENSION 1

AUGUST 2005

TIME ALLOWED: **2 HOURS + 5 MINUTES READING TIME**

WEIGHTING: **40%**

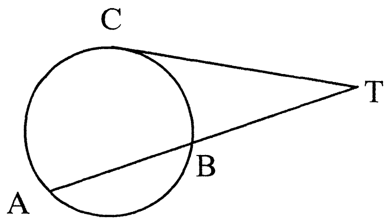
GENERAL INSTRUCTIONS:

- ATTEMPT ALL QUESTIONS.
- USE BLUE/BLACK PEN.
- BOARD APPROVED CALCULATORS MAY BE USED.
- A TABLE OF STANDARD INTEGRALS IS ATTACHED.
- ALL NECESSARY WORKING SHOULD BE SHOWN FOR EACH QUESTION.
- **START EACH QUESTION IN A NEW BOOKLET.**

QUESTION 1 **[12 MARKS]** **[START A NEW BOOKLET]**

a. Solve the inequality $\dfrac{4-x^2}{x}<0$ **[2]**

b.



CT is a tangent to the circle ABC, with ABT a secant intersecting the circle at A and B. Given that CT = 6, BT = 5 find AB. **[2]**

c. The line through the points A(2, −1) and B(4, 1) intersects the line with equation $2y-x+4=0$. Find the acute angle between the lines, to the nearest degree. **[2]**

d. It is known that $(x+1)$ is a factor of the polynomial $P(x)=2x^3-ax+2$. Find the value of a. **[1]**

e. The line $y=kx$ intersects the circle $x^2+y^2-2x-14y+25=0$ at two distinct points. **[3]**

(i) Show that $25(k^2+1)-(7k+1)^2<0$.

(ii) For what value of k is the line $y=kx$ a tangent to the circle?

f. Using the substitution $u=x^2$ find $\int_0^2 x\, e^{x^2}\, dx$, leaving your answer in terms of e. **[2]**

QUESTION 2 **[12 MARKS]** **[START A NEW BOOKLET]**

(a) The function $f(x) = x^3 - \ln(x+1)$ has one root lying between 0.5 and 1. **[4]**

(i) Show that the root lies between 0.8 and 0.9.

(ii) Use one application of Newton's method to find a second approximation to 3 decimal places if $x = 0.85$ is taken as the first approximation.

(b) **[3]**

(i) Show that $\frac{d}{dx}\left(\sin^{-1} x + \cos^{-1} x\right) = 0$ for the domain $0 \le x \le 1$.

(ii) Hence, or otherwise, sketch the graph of $y = \sin^{-1} x + \cos^{-1} x$.

(c) A container of hot water at temperature T°C loses heat when placed in a cooler environment. It cools according to the rule $\frac{dT}{dt} = k\left(T - T_0\right)$, where t is the elapsed time in minutes, T_0 is the environment temperature in degrees (°C). **[5]**

(i) Show that $T = T_0 + Ce^{kt}$, where C a constant, is a solution to the differential equation.

(ii) A container of water at 90°C is placed in a freezer at −20°C. It cools to a temperature of 60°C in 3 minutes. Find the value of k.

(iii) The same container of water, now at 60°C, is then left in an environment at 20°C. Assuming the value of k remains constant, find, to the nearest degree, the temperature after a further 15 minutes.

QUESTION 3 **[12 MARKS]** **[START A NEW BOOKLET]**

(a) The point $P(3,8)$ divides the interval AB externally in the ratio $k : 1$. If A is the point $(0,2)$ and B is the point $(2,6)$, find the value of k. **[3]**

(b) Two points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$. **[6]**

(i) Show that the equation of the tangent to the parabola at Q is $y = qx - aq^2$.

(ii) The tangent at Q and the line through P parallel to the y axis intersect at A. Find the co-ordinates of A.

(iii) Write down the co-ordinates of M, the midpoint of QA.

(iv) Determine the locus of M when PQ is a focal chord.

(c) **[3]**

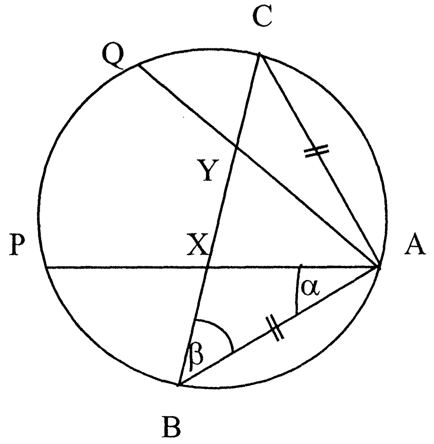
(i) On the same set of axes, sketch the graphs of $y = 2\cos\theta$ and $y = \frac{1}{2}\theta$ for $-\pi \le \theta \le \pi$.

(ii) Use your sketch to find the number of solutions of the equation $2\cos\theta = \frac{1}{2}\theta$ for

$-\pi \le \theta \le \pi$.

QUESTION 4 **[12 MARKS]** **[START A NEW BOOKLET]**

(a)



Let ABPQC be a circle with AB = AC. Also AP intersects BC at X, and AQ intersects BC at Y. Let ∠PAB = α and ∠ABC = β. **[4]**

(i) Copy the diagram into your Answer Book and state why ∠AXC = α + β.

(ii) Show ∠PQB = α.

(iii) Show ∠AQB = β.

(iv) Prove ∠XYQP is a cyclic quadrilateral.

(b) Find all angles θ, where $0 \le \theta \le 2\pi$, for which $\sqrt{3}\sin 2\theta - \cos 2\theta = 1$. **[3]**

(c) **[2]**

(i) A particle moving in a straight line is subject to an acceleration given by $\ddot{x} = -2e^{-x}$ where x is the displacement from the origin in metres. The particle is initially at the origin with a velocity of $2ms^{-1}$. Prove that $v = 2e^{-x/2}$. (You may use the fact that $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x}$).

(ii) Explain the effect on v as x increases without bound.

(d) A particle is moving in simple harmonic motion. It's displacement x at time t is given by $x = 4\sin(2t+3)$. **[3]**

(i) Find the period of the motion.

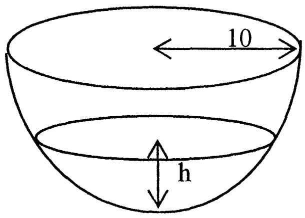
(ii) Find the maximum acceleration of particle.

(iii) Find the speed of the particle when $x = 2$.

QUESTION 5 **[12 MARKS]** **[START A NEW BOOKLET]**

(a) Prove by mathematical induction that $1+3+3^2+3^3+$ ----------- $+3^{n-1}=\dfrac{3^n-1}{2}$ for all $n \geq 1$. **[3]**

(b) **[5]**



(i) The diagram above represents a hemispherical bowl of radius 10cm filled with water to a depth of h cm. By finding the volume generated by rotating a circle $x^2+y^2=100$ between $y=10$ and $y=10-h$ about the y-axis show that the volume of water in the bowl is given by $V=\dfrac{\pi}{3}h^2(30-h)$.

(ii) The hemispherical bowl referred to in **(i)** above is being filled with water at a constant rate of $2\pi\ \text{cm}^3/\text{min}$. At what rate is the depth increasing when the depth of water is 2cm?

(c) **[4]**

(i) Differentiate $x\cos^{-1}x-\sqrt{1-x^2}$ with respect to x.

(ii) Use your result from **c (i)** above to evaluate $\int_0^1 \cos^{-1}x\,dx$

QUESTION 6 **[12 MARKS]** **[START A NEW BOOKLET]**

(a) **[4]**

(i) Solve the equation $2x^3 - 7x^2 - 12x + 45 = 0$, given that two of the roots are equal.

(ii) The equation in **(i)** above has two sets of possible solutions. Explain why only one set of values for the roots is valid.

(b) Find the value of the term that is independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^6$. **[2]**

(c) Let $f(x) = -\left(\frac{x}{1-x^2}\right)$ **[3]**

(i) For what values of x is $f(x)$ undefined?

(ii) Show that $f'(x) < 0$ at all values of x for which the function is defined.

(iii) Hence sketch the curve $y = f(x)$.

(d) **[3]**

(i) Show, by means of a sketch, that the curves $y = x^2$ and $y = -\frac{1}{2}\ln x$ meet at a single point.

(ii) Prove that the tangents to the two curves intersect at right angles at this point of intersection.

Standard Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1; x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x + C, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x \equiv \log_e x, \quad x > 0$

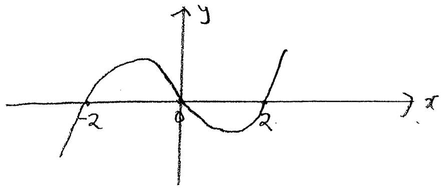
Q1 (12 marks)

(a) $\frac{4-x^2}{x} < 0$

$\frac{x^2(2-x)(2+x)}{x} < 0 \times x^2$ $\frac{1}{2}$

$x(x-2)(x+2) > 0$ $\frac{1}{2}$

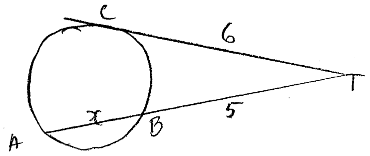
2



1

$\therefore$ Soln $\{-2 < x < 0\}$ and $\{x > 2\}$

(b)



Let $AB = x$

2

$CT^2 = AT.TB$ 1

$6^2 = (5+x)5$

$36 = 25 + 5x$

$5x = 11$

$x = 2{\cdot}2$ cm

(c) Eq. through A, B : $\frac{y-1}{x-4} = \frac{-1-1}{2-4} = \frac{-2}{-2} = 1$ $\frac{1}{2}$

$m_1 = 1$

Eq. line $2y - x + 4 = 0 \Rightarrow y = \frac{1}{2}x - 2$ $\frac{1}{2}$

$\therefore m_2 = \frac{1}{2}$

Angle between lines; $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

2

$= \frac{1-\frac{1}{2}}{1+1(\frac{1}{2})}$

$= \frac{\frac{1}{2}}{\frac{3}{2}}$ 1

$= \frac{1}{3}$

$\therefore \theta = 18^\circ$ (nearest degree)

(d) Since $(x+1)$ is a factor $P(-1) = 0$ $\frac{1}{2}$

1

$P(-1) = 2(-1)^3 - a(-1) + 2$

$= -2 + a + 2$

$= a$ 1/2

$= 0$

$\therefore a = 0$

(e) (i) Since the line intersects the circle

$x^2 + (kx)^2 - 2x - 14(kx) + 25 = 0$ 1/2

$x^2 + k^2x^2 - 2x - 14kx + 25 = 0$

$(1+k^2)x^2 - (2+14k)x + 25 = 0$ $\frac{1}{2}$

Since there are 2 distinct roots $\Delta > 0$

$\therefore b^2 - 4ac > 0$ 1/2

$[-2(1+7k)]^2 - 4(1+k^2)25 > 0$

$4(7k+1)^2 - 4(25)(1+k^2) > 0$ 1/2 1

$(7k+1)^2 - 25(1+k^2) > 0$

ie $25(k^2+1) - (7k+1)^2 < 0$

(ii) The line $y = kx$ is tangential when only one root exists ie $\Delta = 0$

3

$25(k^2+1) - (7k+1)^2 = 0$

$25k^2 + 25 - 49k^2 - 14k - 1 = 0$

$-24k^2 - 14k + 24 = 0$

$12k^2 + 7k - 12 = 0$ 1

$(4k-3)(3k+4) = 0$

$\therefore k = \frac{3}{4}, -\frac{4}{3}$

(f) Let $u = x^2$ when $x = 0, u = 0$

$\therefore du = 2x\,dx$ $x = 2, u = 4$ 1

$\int_0^2 x e^{x^2} dx = \frac{1}{2}\int_0^2 2x e^{x^2} dx$

$= \frac{1}{2}\int_0^4 e^u du$

2

$= \frac{1}{2}\left[e^u\right]_0^4$ 1

$= \frac{1}{2}(e^4 - e^0)$

$= \frac{1}{2}(e^4 - 1)$

Q2.

(a) (i) $f(x) = x^3 - \ln(x+1)$

$f(0.8) = (0.8)^3 - \ln(1.8)$

$= 0.512 - 0.588$

< 0

$f(0.9) = (0.9)^3 - \ln(1.9)$

$= 0.729 - 0.642$

> 0 1

Since $f(0.8)$ and $f(0.9)$ lie on opposite sides of the x axis $\therefore$ a root lies between 0.8 and 0.9 1

(ii) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ 1

4

$f'(x) = 3x^2 - \dfrac{1}{x+1}$ $\qquad f(x) = x^3 - \ln(x+1)$

$f'(0.85) = 3(0.85)^2 - \dfrac{1}{1.85}$ $\qquad f(0.85) = (0.85)^3 - \ln(1.85)$

$= 1.627$ $\qquad = -0.001$

$x_1 = 0.85 - \dfrac{-0.001}{1.627}$ 1

$= 0.851$

(b) (i) $\dfrac{d}{dx}\left(\sin^{-1}x + \cos^{-1}x\right)$

$= \dfrac{1}{\sqrt{1-x^2}} - \dfrac{1}{\sqrt{1-x^2}}$ 1

$= 0$

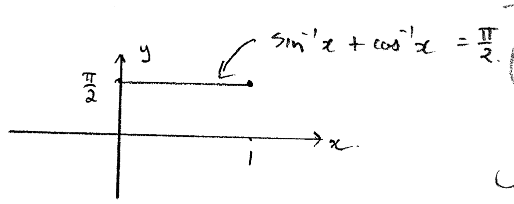
(ii) Since $f'(x) = 0$ $\quad \therefore f(x) = c$, a constant

when $x = 0$, $\quad f(0) = \sin^{-1}0 + \cos^{-1}0$

$= \dfrac{\pi}{2}$ 1

3

$\therefore f(x) = \sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$



1

(c) (i) $T = T_0 + Ce^{kt}$

$\dfrac{dT}{dt} = kCe^{kt}$

$= k(T - T_0)$ 1

$\therefore T = T_0 + Ce^{kt}$ is a soln of $\dfrac{dT}{dt} = k(T - T_0)$

(ii) At $t = 0$, $T = 90$ $\qquad T_0 = -20$

$t = 3$, $T = 60$

$T = T_0 + Ce^{kt}$

$90 = -20 + Ce^0$ 1

$\therefore C = 110$

$T = T_0 + 110e^{kt}$

For $t = 3$, $\quad 60 = -20 + 110e^{3k}$

$80 = 110e^{3k}$

$\therefore e^{3k} = \dfrac{8}{11}$

$3k = \ln\left(\dfrac{8}{11}\right)$ 1

5

$k = \frac{1}{3}\ln\frac{8}{11}$

$= -0.106\ldots$ (3 dp)

(iii) At $t = 0$, $T = 60$ $\qquad T_0 = 20$

$t = 15$, $T = ?$

$T = T_0 + C_1e^{kt}$ 1

$60 = 20 + C_1e^0$

$\therefore C_1 = 40$

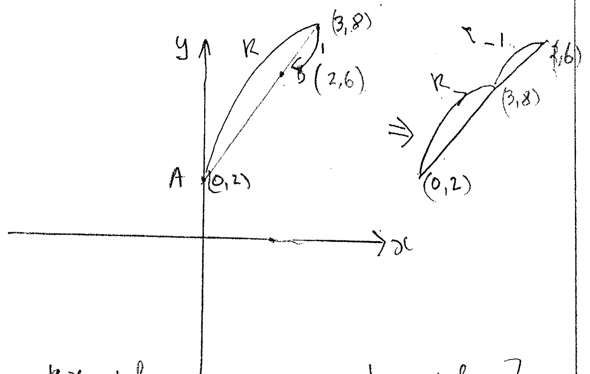
$T = T_0 + 40e^{kt}$

$T = 20 + 40e^{15(-0.106)}$ 1

$= 28°C$

Q3.

(a).



$x_p = \dfrac{kx_2 + lx_1}{k+l}$; $y_p = \dfrac{ky_2 + ly_1}{k+l}$ 1

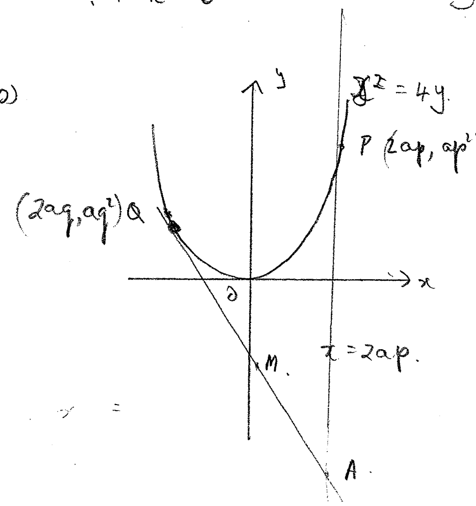
3

$3 = \dfrac{2k - 1(0)}{k-1}$ $\quad 8 = \dfrac{6k - 2(1)}{k-1}$ 1

$3k - 3 = 2k$

$\therefore k = 3$ 1

(b)



(i) $x^2 = 4ay$

$2x = 4ay'$

$y' = \dfrac{2x}{4a} = \dfrac{2.2aq}{4a} = q$ 1

eq. tangent at Q : $\dfrac{y - aq^2}{x - 2aq} = q$

6

$y - aq^2 = qx - 2aq^2$

$y = qx - aq^2$ 1

(ii) Point of intersection A :

$y = qx - aq^2$ and $x = 2ap$.

$\therefore y = q.2ap - aq^2 = 2apq - aq^2$ 1

$\therefore$ A has co-ords $\left[2ap, (2apq - aq^2)\right]$

(iii) Co-ords of M : $\left(\dfrac{2ap + 2aq}{2}, \dfrac{aq^2 + 2apq - aq^2}{2}\right)$

$= \left[a(p+q), apq\right]$ 1

(iii) If PQ is a focal chord $pq = -1$ 1

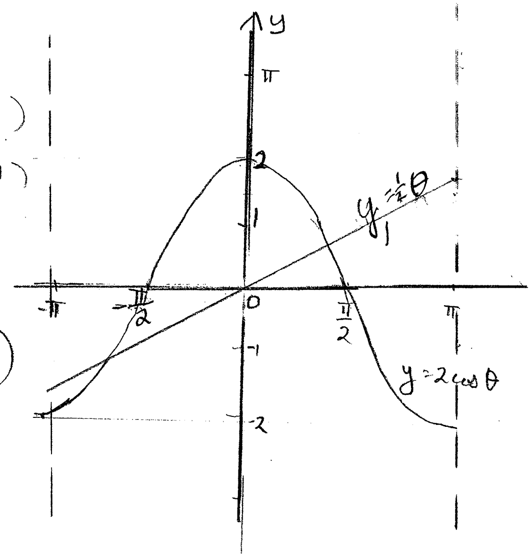
Now, $x = a(p+q)$, $y = apq$

$\therefore y = a(-1) = -a$ 1

Locus of M is $y = -a$

(c)

(i)

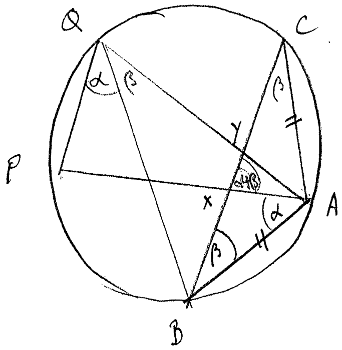


2 (1 each)

3

(ii) No. solutions ; 2 1

Q4.



(i) $\angle AXC = \alpha + \beta$ (ext. $\angle$ of $\triangle AXB$) 1

(ii) $\angle PQB = \angle PAB = \alpha$ (angles at circumfer on same arc PB. 1

4

(iii) $\angle AQB = \angle BCA$ (angles on same arc AB)
$= \angle ABC$ (eq. $\angle$'s, isos $\triangle ABC$) 1
$= \beta$

(iv) $\angle YXA = \angle YQP = \alpha + \beta$ 1
$\therefore$ XYQP is cyclic (ext $\angle$ eqs int opp $\angle$)

b) $\sqrt{3}\sin 2\theta - \cos 2\theta = 1$ $\quad \therefore 0 \le 2\theta \le 4\pi$

$\frac{\sqrt{3}}{2}\sin 2\theta - \frac{1}{2}\cos 2\theta = \frac{1}{2}$ $\quad \sin\alpha = \frac{1}{2}$

$\sin 2\theta\cos\alpha - \cos 2\theta\sin\alpha = \frac{1}{2}$ $\quad \alpha = \frac{\pi}{6}$ 1

$\sin(2\theta - \alpha) = \frac{1}{2}$

3

$\therefore 2\theta - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$ 1

$2\theta = \frac{\pi}{3}, \pi, \frac{14\pi}{6}, \frac{18\pi}{6}$

$= \frac{\pi}{3}, \pi, \frac{7\pi}{3}, 3\pi$ 1

$\theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{7\pi}{6}, \frac{3\pi}{2}$

(c) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2e^{-x}$

$\frac{1}{2}v^2 = -2\int e^{-x}\,dx$

$= \frac{-2e^{-x}}{-1} + c$ 1/2

$\frac{1}{2}v^2 = 2e^{-x} + c$

at $t=0$, $x=0$, $v=2$.

$\therefore \frac{1}{2}\cdot 4 = 2e^{0} + c$

$2 = 2 + c$

$\therefore c = 0$ 1/2

2

$\frac{1}{2}v^2 = 2e^{-x}$

$v^2 = 4e^{-x}$

$v = \sqrt{4e^{-x}}$

$= 2e^{-\frac{x}{2}}$

(ii) as $x \to \infty$, $v \to 2e^{-\frac{\infty}{2}} \to 0$.
as x increases without bound $v \to 0$ 1

(d) $x = 4\sin(2t+3)$ is of form $x = a\sin(nt+\alpha)$

(i) Period $T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi$ sec. 1

(ii) $\ddot{x} = -n^2 x$.

Max acc when $x = a$.

3

$\therefore \ddot{x} = -2^2 \cdot 4 = -16$ m/s^2 1

(iii) $v^2 = n^2(a^2 - x^2)$

$= 2^2(4^2 - 2^2)$

$= 4(12) = 48$ 1

$v = \sqrt{48}$

$= 4\sqrt{3}$ m/s

Q5

(a) assume $1+3+3^2+\dots+3^{n-1} = \frac{3^n-1}{2}$ is true

When $n=1$, LHS $=1$

RHS $= \frac{3^1-1}{2} = 1$ 1

$\therefore$ assumption true for $n=1$

Assume $1+3+3^2+\dots+3^{n-1} = \frac{3^n-1}{2}$ is true for $n=k$

$\therefore 1+3+3^2+\dots+3^{k-1} = \frac{3^k-1}{2}$ is true

For $n=k+1$

$$1+3+3^2+\dots+3^{k-1}+3^k = \frac{3^k-1}{2}+3^k$$

$$= \frac{3^k}{2}+\frac{2.3^k}{2}-\frac{1}{2}$$ 1

3

$$= \frac{3^{k+1}}{2}-\frac{1}{2}$$

$$= \frac{3^{k+1}-1}{2}$$

$\therefore$ statement true for $n=k+1$

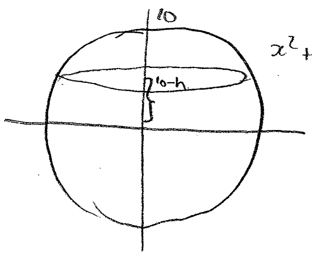
Since statement is true for $n=1$, it is true for $n=2$

Since statement is true for $n=2$, it is true for $n=3$

and so on. 1

$\therefore$ statement is true for all $n \geq 1$

(b)



$$V = \pi\int_{10-h}^{h} x^2\,dy$$

$$= \pi\int_{10-h}^{h}(100-y^2)\,dy$$ 1

5

$$= \pi\left[100y-\frac{y^3}{3}\right]_{10-h}^{10}$$ 1

$$= \pi\left[\left(1000-\frac{10^3}{3}\right)-\left[100(10-h)-\frac{(10-h)^3}{3}\right]\right]$$

$$= \pi\left[1000-1000+100h+\frac{(10-h)^3}{3}-\frac{10^3}{3}\right]$$

$$= \frac{\pi}{3}\left[300h+(10-h)^3-10^3\right]$$

$$= \frac{\pi}{3}\left[300h+1000-300h+30h^2-h^3-1000\right]$$ 1

$$= \frac{\pi}{3}\left[30h^2-h^3\right]$$

$$= \frac{\pi}{3}h^2\left[30-h\right]$$

(ii) $V = \frac{\pi}{3}(30h^2-h^3)$

$$\frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{\pi}{3}(60h-3h^2)\frac{dh}{dt}$$ 1

$$2\pi = \frac{\pi}{3}\left[60(2)-3(2^2)\right]\frac{dh}{dt}$$

$$6 = (120-12)\frac{dh}{dt}$$

$$6 = 108\frac{dh}{dt}$$ 1

$$\therefore \frac{dh}{dt} = \frac{6}{108} = \frac{1}{18}\text{ cm/sec}$$

$$\doteqdot 0.55\text{ mm/sec}$$

(c)(i) $\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$

$$= \frac{d}{dx}\left(x\cos^{-1}x-(1-x^2)^{1/2}\right)$$

$$= \cos^{-1}x-\frac{x}{\sqrt{1-x^2}}-\frac{1}{2}\frac{(-\cancel{2}x)}{\sqrt{1-x^2}}$$ 1

4

$$= \cos^{-1}x-\frac{x}{\sqrt{1-x^2}}+\frac{x}{\sqrt{1-x^2}}$$ 1

$$= \cos^{-1}x.$$

(ii) $\cos^{-1}x = \frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$

$$\int_0^1\cos^{-1}x = \int_0^1\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)$$ 1

$$= \left[x\cos^{-1}x-\sqrt{1-x^2}\right]_0^1$$

$$= (\cos^{-1}1-0)-(0-1)$$ 1

$$= 0+1$$

$$= 1$$

Q6.

(a) (i) Let the roots be α, α, β.

sum of roots $\alpha+\alpha+\beta = -\frac{b}{a}$

product pairs $\alpha\alpha + \alpha\beta + \alpha\beta = \frac{c}{a}$ ½

product of roots $\alpha\alpha\beta = -\frac{d}{a}$

$2\alpha + \beta = \frac{7}{2}$ ½

$\alpha^2 + 2\alpha\beta = -\frac{12}{2} = -6$ 1

$\alpha^2\beta = -\frac{45}{2}$

From ① $\beta = \frac{7}{2} - 2\alpha$

From ② $\alpha^2 + 2\alpha\left(\frac{7}{2} - 2\alpha\right) = -6$

$\alpha^2 + 7\alpha - 4\alpha^2 = -6$

$3\alpha^2 - 7\alpha - 6 = 0$ 1

$(3\alpha + 2)(\alpha - 3) = 0$

4

$\therefore \alpha = 3, -\frac{2}{3}$

$\beta = -\frac{5}{2}, \frac{29}{6}$ 1

$\therefore$ possible solns are $3, 3, -\frac{5}{2}$ and $-\frac{2}{3}, -\frac{2}{3}, \frac{29}{6}$

(ii) Substitution of $-\frac{2}{3}, -\frac{2}{3}, \frac{29}{6}$ in $\alpha^2\beta > 0$ not $\frac{-45}{2}$ 1

$\therefore$ only roots are $3, 3, -\frac{5}{2}$

(b) $T_{k+1} = {}^6C_k (x^2)^{6-k} (-2x^{-1})^k$

$= {}^6C_k\, x^{12-2k} (-2)^k x^{-k}$ 1

2

$= {}^6C_k (-2^k)\, x^{12-3k}$

Term independent when $12 - 3k = 0$ 1

$\therefore k = 4$

$\therefore T_5$ required $= {}^6C_4 (-2)^4 = 15 \times 16 = 240$

(c) $f(x) = -\left(\frac{x}{1-x^2}\right) = \frac{-x}{(1+x)(1-x)}$

(i) Since $(1-x)(1+x) \neq 0$ 1

3

$\therefore x$ undefined at $x = 1, -1$

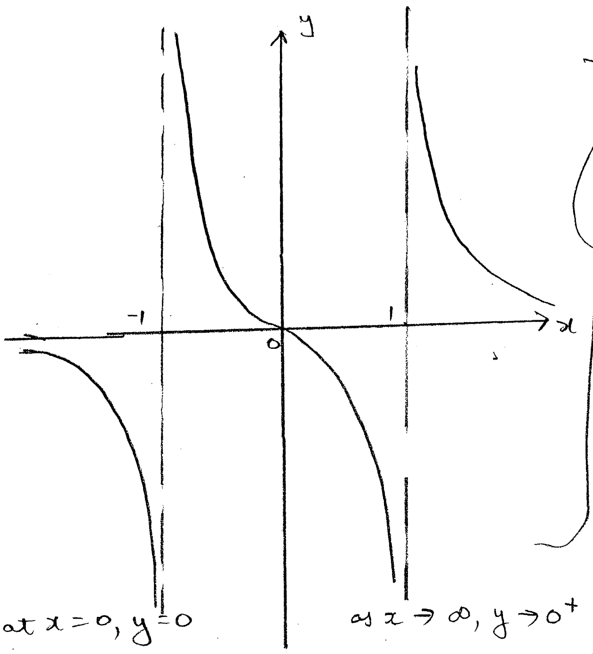
(ii) $f(x) = \frac{-x}{1-x^2}$

$f'(x) = \frac{(1-x^2)(-1) - (-x)(-2x)}{(1-x^2)^2}$

$= \frac{-1 + x^2 - 2x^2}{(1-x^2)^2}$ 1

$= \frac{-(1+x^2)}{(1-x^2)^2}$

< 0 since $1+x^2, (1-x^2)^2 > 0$ for all $x, x \neq -1, 1$



1

at $x = 0, y = 0$

as $x \to 1^+, y \to \infty$

as $x \to 1^-, y \to -\infty$

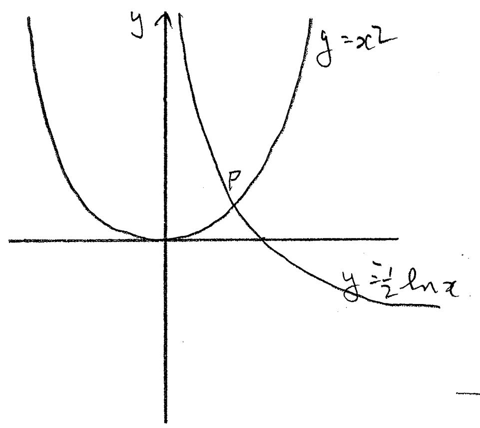
as $x \to -1^+, y \to \infty$

as $x \to -1^-, y \to -\infty$

as $x \to \infty, y \to 0^+$

as $x \to -\infty, y \to 0^-$

(d) (i)



1

3

(ii) $y_1 = x^2$ $y_2 = -\frac{1}{2}\ln x$ 1

$y_1' = 2x$ $y_2' = -\frac{1}{2x}$

Let P have abscissa $x = a$

$\therefore m_1 = 2a$ $m_2 = -\frac{1}{2a}$

$m_1 m_2 = 2a \times -\frac{1}{2a} = -1$ 1

$\therefore$ tangents are perp. at P.

Q7. a 4 b 8

(a). (i)

$(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots {}^{2n}C_r x^r + \dots {}^{2n}C_{2n} x^{2n}$

$= 1 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_r x^r + \dots + x^{2n}$ 1

(ii) When $x = 1$

$2^{2n} = {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \dots {}^{2n}C_n + \dots {}^{2n}C_{2n-1} + {}^{2n}C_{2n}$

$= \left[{}^{2n}C_0 + {}^{2n}C_1 + \dots + {}^{2n}C_n\right] - {}^{2n}C_n + \left[{}^{2n}C_n + \dots + {}^{2n}C_{2n}\right]$

$= 2\left[{}^{2n}C_0 + {}^{2n}C_1 + \dots {}^{2n}C_n\right] - {}^{2n}C_n$

$\therefore 2\left[{}^{2n}C_0 + {}^{2n}C_1 + \dots + {}^{2n}C_n\right] = 2^{2n} - {}^{2n}C_n$ 1

$2 \sum_{r=0}^{n} {}^{2n}C_r = 2^{2n} - {}^{2n}C_n$

4

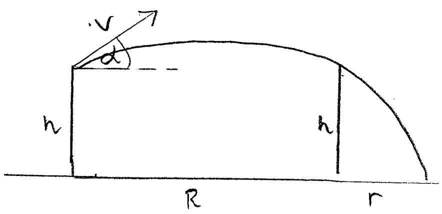
$\sum_{r=0}^{n} {}^{2n}C_r = 2^{2n-1} - \frac{1}{2}\, {}^{2n}C_n$

$= 2^{2n-1} - \frac{1}{2} \cdot \frac{(2n)!}{(n)!\,(n!)}$

$= 2^{n-1} - \frac{(2n)!}{2(n!)^2}$ 1

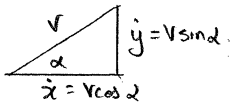
(b)

(i)



8

Horizontally



$\ddot{x} = 0$

$\therefore \dot{x} = \int 0\,dt = C_1$

at $t = 0$, $\dot{x} = V\cos\alpha$

$x = \int \dot{x}\,dt = \int (V\cos\alpha)\,dt = Vt\cos\alpha + C_2$ 1

at $t = 0, x = 0 \quad \therefore C_2 = 0$

$x = Vt\cos\alpha$

Vertically $\ddot{y} = -g$

$\dot{y} = -\int g\,dt = -gt + C_1$

at $t = 0$, $\dot{y} = V\sin\alpha$

$\therefore C_1 = V\sin\alpha$

$\dot{y} = -gt + V\sin\alpha$

$y = \int \dot{y}\,dt = \int(-gt + V\sin\alpha)\,dt$ 1

$= -\frac{1}{2}gt^2 + Vt\sin\alpha + C_2$

at $t = 0$, $y = h$. $\therefore h = C_2$

$y = -\frac{1}{2}gt^2 + Vt\sin\alpha + h$

(ii) From (i), $t = \frac{x}{V\cos\alpha}$

$y = -\frac{1}{2}g\left(\frac{x}{V\cos\alpha}\right)^2 + V\left(\frac{x}{V\cos\alpha}\right)\sin\alpha + h$

$= -\frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha} + x\tan\alpha + h$ 1

$y = h + x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha}$

(iii) The ball clears the fence when $y \geqslant h, x \geqslant R$

$\therefore \quad h + R\tan\alpha - \frac{gR^2}{2V^2\cos^2\alpha} \geqslant h$

$R\tan\alpha - \frac{gR^2}{2V^2\cos^2\alpha} \geqslant 0$

$\frac{R^2 g}{2V^2\cos^2\alpha} \leqslant R\tan\alpha$

$\frac{2V^2\cos^2\alpha}{R^2 g} \geqslant \frac{1}{R\tan\alpha}$ 2

$V^2 \geqslant \frac{R^2 g}{(R\tan\alpha)(2\cos^2\alpha)}$

$V^2 \geqslant \frac{gR}{2\tan\alpha\cos^2\alpha}$

$V^2 \geqslant \frac{gR}{2\sin\alpha\cos\alpha}$

(iv) at C, $y = 0$, $x = R + r$

$\therefore 0 = h + (R+r)\tan\alpha - \frac{(R+r)^2 g}{2V^2\cos^2\alpha}$ $\frac{1}{2}$

$\frac{(R+r)^2 g}{2V^2\cos^2\alpha} = (R+r)\tan\alpha + h$

$\frac{g}{2\cos^2\alpha} = V^2\left[\frac{\tan\alpha}{(R+r)} + \frac{h}{(R+r)^2}\right]$

$\geqslant \frac{gR}{2\sin\alpha\cos\alpha}\left[\frac{\tan\alpha}{(R+r)} + \frac{h}{(R+r)^2}\right]$

$\frac{1}{\cos\alpha} \geqslant \frac{R}{\sin\alpha}\left[\frac{\tan\alpha}{(R+r)} + \frac{h}{(R+r)^2}\right]$

$$\therefore \frac{\sin\alpha}{\cos\alpha} \geq \frac{R\tan\alpha}{(R+r)} + \frac{Rh}{(R+r)^2}.$$

$$\tan\alpha - \frac{R\tan\alpha}{R+r} \geq \frac{Rh}{(R+r)^2}.$$

$$\tan\alpha\left(1 - \frac{R}{R+r}\right) \geq \frac{Rh}{(R+r)^2}.$$

$$\tan\alpha\left[\frac{R+r-R}{R+r}\right] \geq \frac{Rh}{(R+r)^2}.$$

$$\tan\alpha\left(\frac{r}{R+r}\right) \geq \frac{Rh}{(R+r)^2}.$$

$$\tan\alpha \geq \frac{Rh}{r(R+r)}$$

1

(v) $V^2 \geq \frac{gR}{2\cos\alpha\sin\alpha}.$

$$2\sin\alpha\cos\alpha \geq \frac{gR}{V^2}.$$

$$\sin 2\alpha \geq \frac{gR}{V^2}$$

$$\geq \frac{10(80)}{50^2} = \frac{800}{2500} = 0.32.$$

1

$$\therefore 2\alpha \geq 18.66^\circ, 161.34^\circ$$

$$\alpha \geq 9.33^\circ, 80.67^\circ$$

Since closest distance required

$\alpha = 80.67^\circ$

$$\tan\alpha \geq \frac{Rh}{r(R+r)}$$

$$\tan 80.67^\circ \geq \frac{80(1)}{r(80+r)}$$

$$r^2 + 80r \geq \frac{80}{\tan 80.67}$$

$$r^2 + 80r - 13.14 \geq 0.$$

$$r = \frac{-80 \pm \sqrt{(-80)^2 - 4(1)(-13.14)}}{2}$$

$$= \frac{-80 \pm 80.32}{2}$$

$$= \frac{0.32}{2}, \frac{-160.32}{2}$$

$$= 0.16, -80.16$$

1

Since $r > 0$, $r = 0.16$ m ie 16 cm beyond B.